NONLINEAR PROBLEMS ON THE DEFORMATION OF ELASTIC BODIES BY A MAGNETIC FIELD

PMM Vol. 34, №4, 1970, pp. 653-671
K. Sh. KHODZHAEV
(Leningrad)
(Received January 8, 1970)

The formulation of "elastically linear" problems of the nonlinear theory of magnetoelasticity is described. Problems in magnetoelasticity are understood to be problems of determining the magnetic field in a domain enclosing elastic bodies, and the strain state of these bodies under the effect of ponderomotive forces. Situations are examined when assumptions of linear elasticity theory are acceptable, but it is necessary to take account of the dependence of the field on the displacements. Two classes of problems encompassing, respectively, the equilibrium of ferromagnetic bodies and conductors with currents at distances commensurate with the elastic displacements, are separated out. As illustrations, the bending of a ferromagnetic membrane and a string-strip by a magnet and the equilibrium of flexible conductors are examined. For a circular membrane the problem reduces to the case of the Emden-Fowler equation which has not been solved by study. It is shown that the system can have arbitrarily, and even infinitely, many equilibrium modes. The problem of the equilibrium of linear conductors reduced to determining the closed and self-intersecting trajectories of a point in a central force field whose magnitude is inversely proportional to the distance. Plane equilibrium modes are found. A number of other boundary value problems of the theory are formulated.

1. Elastically linear problems of nonlinear magnetoelasticity theory. Let us consider a set of elastic bodies in a magnetic field. As usual in electrodynamics, let the field intensity of the foreign electromotive forces be considered known; it can be given as a function of either the space coordinates or of the elastic element. We also consider the permeability μ and the conductivity σ in the undeformed state to be known functions of the coordinates. Let us consider magnetically linear media (*). Let us take the density of the volume forces $\mathbf{f}$ as

$$\mathbf{f} = \mathbf{j} \times \mathbf{B} - {}^1/_2 H^2 \operatorname{grad} \mu \tag{1.1}$$

where $\mathbf{j}$ is the current density, $\mathbf{B}$ the induction, and $\mathbf{H}$ the magnetic field intensity. The density of the surface forces on surfaces of discontinuity of μ and the density of the surface or line forces in cases when surface or line currents are examined are determined from (1.1) by passing to the limit. The remaining ponderomotive forces and strictive effects are not taken into account. Let us limit ourselves to equilibrium problems. It is necessary to find $\mathbf{j}$, $\mathbf{B}$ and the vector of the elastic displacements $\mathbf{u}$ caused by the forces $\mathbf{f}$. These vector fields must be determined jointly. Indeed, $\mathbf{j}$ and $\mathbf{B}$ depend on $\mathbf{u}$ since the displacements affect the distribution of μ and the conductivity σ in space, and $\mathbf{u}$ is determined by the forces $\mathbf{f}$ dependent on $\mathbf{j}$ and $\mathbf{B}$.

Let us consider the displacements small, and let us accept the assumptions of linear elasticity theory. Let us describe two situations when it is necessary to take account of

*) Only ferromagnets can have values μ needed later, and they are therefore considered as magnetically linear materials with high permeability.

the dependence of the field on the displacements even under these assumptions. The former occurs when the elastic system includes closely spaced ferromagnetic bodies, and the second when it contains conductors with currents in a strongly inhomogeneous field. Moreover, certain conditions should still be satisfied; they will be enumerated below.

Let there be bodies in space so that the spacings between some sections of their surfaces are small (in the sense indicated below). Let us assume that all three dimensions of these bodies are of the same order of magnitude, and both the dimensions of the closely disposed surface sections are commensurate with the characteristic dimension of the bodies. The closely packed sections should hence be of a "suitable" shape (so that the surfaces could "abut" each other in a domain of significant dimensions), for example, they may be plane. Let us also consider at least one of the closely-packed surface sections not to be fixed.

The following notation is taken for the characteristic values: l are the dimensions of the bodies, Δ_0 the spacing between the closely packed sections measured along the normal to one of the surfaces, a the elastic displacements, μ_1 and μ_0 the permeabilities of the bodies and the surrounding inelastic medium. The ratio a / l is considered small; its magnitude is the criterion of smallness of the displacements.

The following conditions should be satisfied in the situation described: $\Delta_0 / l = O(a / l)$, i.e. the displacements, including the relative displacements of the surfaces along their normals, are commensurate with the spacings between the bodies; $\mu_0 / \mu_1 = O(\Delta_0 / l)$, i.e. the ratio of the permeabilities of the ambient medium and the bodies is small (larger values of μ_1 are also admissible when $\mu_0 / \mu_1 \sim (\Delta_0 / l)^2$, etc.); a permeability value of the order of μ_1 is achieved at spacings of the order of Δ_0 or less from the surfaces of the bodies; the characteristic induction B_0 in the domain between abutting surfaces is one order greater than the induction B_2 in the ambient medium at spacings of the order of l from the surfaces of the bodies, i.e. $B_2 / B_0 = O(\Delta_0/l)$.

This latter assumption imposes a constraint on the shape of the bodies and the configuration of the currents. They should be such that the "number" of lines of induction which are closed on the path "point inside the body–space between abutting surfaces – another body – another space, etc. – original point inside the first body" would exceed the number of lines traversing a distance of the order of l outside the bodies. The characteristic induction B_1 within the bodies is commensurate with B_0.

Under the assumption made, the induction $\mathbf{B}$, and therefore the forces $\mathbf{f}$ as well, will depend essentially on the displacements. Indeed, let us examine two contours C_* and C passing through the same elastic points, the former in undeformed, and the latter in deformed systems, and being closed just through the bodies and the narrow gaps between them. Let S_* and S be surfaces based on C_* and C, and n_* and n their normals. From the relationships

$$\oint_{C_*} \mathbf{H}_* d\mathbf{C}_* = \int_{S_*} \mathbf{j}_* \mathbf{n}_* dS_*, \qquad \oint_{C} \mathbf{H} d\mathbf{C} = \int_{S} \mathbf{j}\mathbf{n} dS \tag{1.2}$$

the approximate equalities follow

$$B_{0*}\Delta_0\mu_0^{-1} + B_{1*}l\mu_1^{-1} = I_*, \qquad B_0(\Delta_0 + a)\mu_0^{-1} + B_1 l\mu_1^{-1} = I \tag{1.3}$$

Here and henceforth the quantities with asterisk refer to the undeformed and without asterisk to the deformed states; I, I_* are the characteristic total currents. Proceeding from the estimates taken for Δ_0, a, l, μ_0 and μ_1, the order of variation in the induction during deformation can now be estimated

$$\Delta B = B_0 - B_{0*} \sim (a \,/\, \Delta_0) B_0 \tag{1.4}$$

when the total current is unchanged. It is seen from (1.4) that $\Delta B \,/\, B_{0*} = O\,(1)$.

A change in current configuration also yields some contribution to ΔB but even if it is neglected, the magnetoelastic problem will be "coupled" and nonlinear. However, if $\Delta_0 / l = O\,(1)$ or $\mu_0 / \mu_1 = O\,(1)$, and the currents "are diverse" at a spacing of the order of l (see later), then the currents, field and forces f can be sought for the undeformed state. The problem then decomposes into a problem of stationary current distribution, a problem of magnetostatics, and a problem of elasticity theory solved in sequence. The solution is hence found to the accuracy of higher terms in $a \,/\, l$, which are discarded anyway in linear elasticity theory.

In the case of magnetically nonlinear media, formulation of the problem is retained except that μ_1 must be understood to be the characteristic value of $dB \,/\, dH$ in the material. Moreover, the requirement that the zones where H is so great that $dB \,/\, dH = O\,(\mu_0)$ did not alter the distribution of the lines of induction described above must be satisfied. This will occur, say, if the sizes of these zones are of the order of Δ_0.

As has been mentioned, $|\,\mathrm{grad}\,\mu\,|$ near the surfaces of the bodies should be sufficiently large. Hence, to the accuracy of the highest members in a/l the forces $H^2\,\mathrm{grad}\,\mu$ acting in these domains can be considered as a surface loading even if μ grows continuously in the body from $\mu = \mu_0$ to $\mu = O\,(\mu_1)$.

The problem can refer not only to a set of bodies, but also to one elastic body which should, however, have a specific outline. (For example, a ferromagnetic torus with a narrow cutout bounded by planes normal to the torus axis. The loading will be applied to the cutout boundaries).

The following changes in the fundamental situation described above are admissible. Besides the case $B_2 \,/\, B_0 = O\,(\Delta_0 \,/\, l)$ the case $B_2 \,/\, B_0 = O\,(1)$ is also possible, but it should be obtained from the preceding by "superposition" of a field on the whole system, whose characteristic induction B_2 both near to and far from the bodies would be commensurate with the induction B_0 available earlier. (Currents of one order of magnitude greater than the initial currents are needed to produce this field). The case, however, when $B_2 \,/\, B_0 = O\,(1)$ and all the lines of induction traverse a path of the order of l outside the body is excluded. Also admitted is the case when $B_1 \gg B_0$ owing to the introduction of an additional field, almost all of whose lines of force are entirely within a single body. Finally, bodies with radically different dimensions in the three directions can be considered. Furthermore, only those cases are examined when a, $\Delta_0 \ll b$, where b is the least dimension of the closely packed surfaces commensurate with the "second" body dimension. But the third body dimension h can be on the order of a, for example, in rod flexure. Then another assumption ought to be made on the smallness of the ratio $\mu_0 \,/\, \mu_1$, i.e. consider that $\mu_0 \,/\, \mu_1 = O\,(bh \,/\, l^2)$, etc., where l is the greatest characteristic dimension of the body.

Let us examine the second fundamental situation in which the forces depend on the displacements. It is here required that the loading be produced by the effect of a strongly inhomogeneous field on the current so that, for example, $a\,|\,\mathrm{grad}\,B\,| \sim B$. Let volume or surface currents flow in a body with a sufficiently smooth surface. Let points of the body be displaced by an amount of the order of a which is small compared with the body sizes. Then, in all space including the domain near the body surface, changes in the field will be small compared with the initial values. It hence follows that the field should be produced by currents flowing either in a body, at least two of whose dimensions Δ_0 are commensurate with the displacements, or in a body with a strongly curved surface (having domains where one of the curvatures is $k = O(1/a)$). The first case results in the problem of equilibrium of a conductor of a slender rod type either near another similar conductor, or near the surface of a ferromagnet.

The formulation described above is sufficiently natural. Indeed, if it is required to deform an elastic system by forces $\mathbf{f}$, then ferromagnets or conductors with currents should logically be placed at distances approximately equal to the required elastic displacements. Values of l/a comparable to the relative permeabilities μ_1/μ_0 of the ferromagnets, or even one order less, are possible in the elastic domain.

2. Equilibrium of ferromagnets. Nonlinear boundary value problems. Only the lowest terms in a/l are retained in formulating the equations of linear elasticity theory. It is hence natural to retain only the lowest terms in the determination of $\mathbf{j}$ and $\mathbf{B}$. Furthermore, some cases are examined when the field equations can be integrated in this approximation, and the ponderomotive forces can be expressed in terms of displacements. We will hence arrive at a nonlinear boundary value problem for just $\mathbf{u}$.

On determining the currents. Let us assume that the dependence of the conductivity on the deformation, the Hall currents, etc. can be neglected, and reasons for the origination of nonlinearities different from those mentioned in Sect. 1 are not taken into account (for example, the influence of displacements on currents owing to the change in resistivity of the medium in the domain between closely packed surfaces). Let the foreign electric field intensity be given as a function of an elastic point. Then the current density can be determined in conformity with the relationship $\mathbf{j}(\mathbf{r} + \mathbf{u}(\mathbf{r})) = \mathbf{j}_*(\mathbf{r})$, where $\mathbf{r}$ is the coordinate vector of a point of a body in the undeformed state, $\mathbf{u}$ is its displacement neglecting particularly changes in the orientation of $\mathbf{j}$ due to elastic rotations. The currents thereby turn out to be expressed in terms of the displacements. Moreover, the dependence on the displacements should be taken into account only for currents interacting with other nearby currents to which the distance changes substantially during deformation (according to Sect. 1). For the remaining currents it can be assumed that $\mathbf{j}(\mathbf{r}) = \mathbf{j}_*(\mathbf{r})$.

Therefore, a magnetostatic problem is obtained to seek $\mathbf{B}$, where the equations and boundary conditions depend on $\mathbf{u}$ in a known manner. For its solution in the problem of equilibrium of ferromagnets (see Sect. 5 for the equilibrium of closely packed conductors with currents) and under the condition $B_2 / B_0 = O(\Delta_0/l)$ it can be considered that the field is localized within the bodies and in the gaps between the closely packed surfaces, i.e. $B = 0$ outside these domains. The field in the gaps between the bodies can be found thus to the accuracy mentioned in a number of cases.

Let us consider a thin layer between pieces of surfaces σ_1 and σ_2 with a given potential $\varphi_1(M_1)$, $\varphi_2(M_2)$, where M_1 is a point on σ_1 and M_2 on σ_2. Let two dimensions of the layer be mutually commensurate (let l denote the appropriate characteristic spacing), and the third dimension is the spacing between σ_1 and σ_2 (its characteristic value is denoted by Δ_0) which is small compared to l. We consider the surfaces sufficiently smooth, and their curvatures of the order of $1/l$ or less. Let $\mathbf{n}_1$ denote the unit vector of the normal to σ_1 directed into the layer, and Δ_1 the length of its segment between σ_1 and σ_2. For all M_1 the ratio $\Delta_1(M_1)/l$ is on the order of Δ_0/l. If the normal to σ_1 passes through the points M_1 and M_2 on σ_1 and σ_2 and $\mathbf{n}_2$ is the unit vector of the normal to σ_2 directed outward from M_2, then the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is of the order of Δ_0/l; to this accuracy $\mathbf{n}_1$ and $\mathbf{n}_2$ may not be distinguished. Let $|\operatorname{grad}\varphi_{1,2}|$ be "not large", i.e. the characteristic values are $|\operatorname{grad}\varphi_{1,2}| = O(\delta\varphi/l)$, where $\delta\varphi$ is the characteristic value of $|\varphi_1 - \varphi_2|$. Then the solution of the Dirichlet problem (the values of φ outside σ_1, σ_2 are unessential)

$$\Delta\varphi = 0, \qquad \varphi = \varphi_1 \text{ on } \sigma_1, \ \varphi = \varphi_2 \text{ on } \sigma_2 \tag{2.1}$$

everywhere within the layer, except in domains of size $O(\Delta_0)$ near the edges is

$$\varphi(M_1, \xi) = \frac{\varphi_2(M_2) - \varphi_1(M_1)}{\Delta_1(M_1)}\xi + \varphi_1(M_1) + \Delta_0/l\ldots \tag{2.2}$$

Here a point within the layer, which lies on the normal to σ_1 passing through $M_1 \in \sigma_1$ and $M_2 \in \sigma_2$ is determined by the coordinates of the point M_1 and the distance ξ to σ_1; $0 \leqslant \xi \leqslant \Delta_1$. If φ is the scalar magnetic potential, $\mathbf{j} = 0$ and $\mu = \mu_0 = \text{const}$ within the layer, σ_1 and σ_2 are surfaces of discontinuity of μ, then the field intensity in the layer $\mathbf{H}_0$ and the surface loading applied, say, to σ_1

$$\mathbf{H}_0 = \operatorname{grad}\varphi = -\frac{\varphi_2(M_2) - \varphi_1(M_1)}{\Delta_1(M_1)}\mathbf{n}_1 + \ldots, \ \mathbf{q}_1(M_1) = \\ = \frac{1}{2}\mu_0\left[\frac{\varphi_2(M_2) - \varphi_1(M_1)}{\Delta_1(M_1)}\right]^2\mathbf{n}_1 + \ldots \tag{2.3}$$

are found from (2.2) (terms of order μ_0/μ_1 are naturally discarded in calculating the load according to (1.1)). These relationships can even be utilized when σ_1 and σ_2 have singularities of edge, tooth, etc., type, and $\Delta_1(M_1)$ is discontinuous. Both $\mathbf{H}_0$ and $\mathbf{q}_1$ are determined everywhere from (2.3) except in domains with dimensions $O(\Delta_0)$ near the edges and discontinuities. The contribution of these domains to $\mathbf{q}_1$ can be ignored. The relationships (2.3) are generally inapplicable if the number of discontinuities is comparable to l/Δ_0 (such systems are encountered in engineering).

From the expression for the field energy in the layer

$$W_0 = \frac{1}{2}\int_V \mathbf{BH}\,dV = \frac{1}{2}\int_{V_1} \mu_0 \operatorname{grad}^2\varphi\,dV = \frac{1}{2}\mu_0\int_{\sigma_1} \frac{(\varphi_2 - \varphi_1)^2}{\Delta_1}\,d\sigma_1 + \ldots \tag{2.4}$$

it follows that the forces $\mathbf{q}_1$ can be determined by variating just W_0; a variation of the remaining part of the total field energy yields a contribution of the form $(\mu_0/\mu_1)\mathbf{q}_1$. But the field energy in the substance W_1 is itself comparable to W_0 in the general case, as is seen from the estimates $W_0 \sim \Delta_0 l^2 B_0^2/\mu_0$ and $W_1 \sim l^3 B_1^2/\mu_1$.

The following might be an illustration for the relationships obtained. Let σ_1, σ_2 be identical plane figures of area S and $\varphi_2 - \varphi_1 = \text{const}$. Let us find the force Q_1 attracting σ_1 and σ_2

$$Q_1 = \int_{\sigma_1} \mathbf{q}_1\mathbf{n}_1\,d\sigma_1 = \frac{1}{2\mu_0}B_0^2 S$$

where B_0 is the induction in the layer. This is the known approximate Maxwell formula.

The field between the bodies is expressed by the relationships (2.2) and (2.3) in terms of the scalar potential on their surfaces and displacements. The potential however should be found from a single problem to determine the field between and within the bodies as a function of the displacements. Possibilities for simplification may be present here; let us examine some of them.

In general, both the magnitudes and the directions of the vectors $\mathbf{B}$ within the bodies depend essentially on $\mathbf{u}$. Cases are also possible when it is admissible to consider only the magnitudes, but not the directions of $\mathbf{B}$, to vary in the material excepting, perhaps, in domains whose contribution to the field energy can be ignored. To do this it is required that one of the dimensions of the bodies l considerably exceed the two other dimensions b and h. Although different situations are possible here; for example, depending on whether the orders of l or b should be considered as regards the dimension of closely packed pieces of surface, in any case the length of a portion of the line of induction within the body will be comparable, for "almost all" lines, to l and can be found at once. Moreover, the distribution of $\mathbf{B}$ in "normal" cross sections can usually be determined sufficiently accurately in bodies of such shape. This permits relating $\mathbf{u}$ to $\varphi_2 - \varphi_1$ by using (1.2). Appending the relations obtained to the equations of elasticity theory, we obtain a nonlinear system in the displacements and the scalar potential (see example in Sect. 3).

Simpler than the others is the case when both dimensions of the closely packed sections are on the order of b, and the length of segments of the lines of force in the substance is on the order of l. Then to the accuracy accepted φ_1, φ_2 can be considered independent of M_1, M_2. We hence arrive at a boundary value problem for just $\mathbf{u}$. However, values of the potentials of abutting surfaces should be determined during the solution; they depend on the displacements. Let us now assume that l diminishes in the system corresponding to this last case while the other dimensions are retained. When l becomes commensurate with b, a system is obtained for which it should be considered that $\mu = \infty$ within the body. The determination of the scalar potentials is hence substantially simplified.

Let us form the equilibrium equations of several specific systems. Let us initially assume $\mu = \infty$ within the bodies, and then (in Sect. 3), let us discuss how to take account of the imperfection of a ferromagnet.

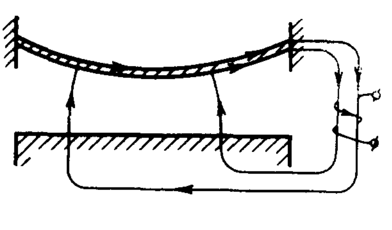

Fig. 1

First let us consider the problem of bending of a ferromagnetic membrane by an electromagnet. Let us consider all the lines of induction to be closed in conformity with Fig. 1 and to be enclosed by the same total current I. Since it was assumed $\mu = \infty$ on their whole length except in the gap between the magnet and the membrane, the shape of the magnetic circuit between the membrane and the abutting surface of the ferromagnet is unessential. It is assumed that this surface is plane and nondeformable, and its contour duplicates the membrane contour Γ' (or encloses it). The difference between the scalar potentials of the membrane φ_1 and the adjoining surface will be $\varphi_1 - \varphi_2 = -I$. Let Δ_0 denote the spacing between the membrane and the magnet surface in the equilibrium position, u the membrane deflection, and T the tension per unit length. In determining the loading in (2.3) it is necessary to take $\Delta_1 = \Delta_0 - u$. Let us set $v = u/\Delta_0$, $\varkappa^2 = \mu_0 I^2 / 2T\Delta_0^3$ and let us introduce dimensionless coordinates in the domain Γ' occupied by the membrane, obtained by multiplying the corresponding dimensional coordinate by $\varkappa$. Referring the loading to the nondeformed membrane, as in linear elasticity theory, we obtain the equilibrium equation

$$\Delta v + \frac{1}{(1-v)^2} = 0 \tag{2.5}$$

The boundary condition will be $v/_\Gamma = 0$; the Laplace operator and the curve Γ are given on planes of the dimensionless coordinates. The one-dimensional analog of (2.5)

$$\ddot{v} + \frac{1}{(1-v)^2} = 0 \tag{2.6}$$

describes the equilibrium of a ferromagnetic string-strip, i.e. a stretched tape whose width b is many times less than its length l but much greater than its deflection u. The dots in (2.6) denote differentiation with respect to τ, $v = v(\tau)$, $\tau = \varkappa x$; here x is the coordinate measured along the string, $\varkappa = b\mu_0 I^2/2T_1\Delta_0^3$, and T_1 is the tension.

If the ferromagnet surface abutting the membrane or string is not plane, then in place of (2.5) and (2.6) we obtain, respectively,

$$\Delta v + \frac{1}{(f-v)^2} \qquad \ddot{v} + \frac{1}{(f-v)^2} = 0 \tag{2.7}$$

where $f = \Delta_{1*}/\Delta_0$ is a known function of the point, Δ_{1*} the spacing between points of the membrane (string) and the magnet in the equilibrium position, and Δ_0 is any constant.

The equation for bending of a thin ferromagnetic plate is formed analogously

$$\Delta\Delta v - \frac{1}{(1-v)^2} = 0 \tag{2.8}$$

Its one-dimensional analog

$$v^{\mathrm{IV}} - \frac{1}{(1-v)^2} = 0 \tag{2.9}$$

describes bending of a beam. For a nonplanar magnet surface these equations are transformed to a form analogous to (2.7). Equations for variable thickness plates and an inhomogeneous beam can also be written down. Hence, if the lower surface of the plate or beam is nonplanar, then both the first (elastic) and the second (magnetic) terms change in (2.8) and (2.9). In general, a loading of the form (2.3) can be applied to bodies of diverse shapes, which generates many nonlinear boundary value problems on the equilibrium of perfect ferromagnetic elastic bodies. In cases when some given loading acts on the body in addition to electromagnetic forces, problems occur on the "interaction" between these two factors. Making the substitution $v_* = 1 - f + v$ in (2.7), we obtain

$$\Delta v_* + \frac{1}{(1-v_*)^2} = -\Delta f \tag{2.10}$$

from which it follows that curvature of the magnet is equivalent, in a specific sense, to application of a given external loading.

Equations of the type (2.5) – (2.9) can evidently be obtained from the variational principle $\delta(\Pi - W_0) = 0$, where Π is the potential energy. It is useful in stability investigations in particular.

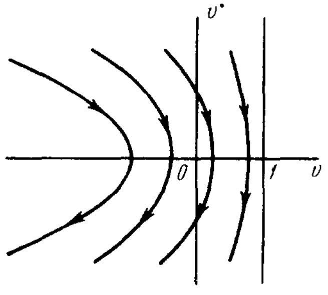

Fig. 2

3. Bending of a ferromagnetic string-strip. Equilibrium curve and its stability. Let us examine a clamped string. The boundary conditions in dimensionless variables will be $v(0) = v(\varkappa l) = 0$. Equation (2.6) admits of the first integral

$v'^2 + (1 - v)^{-1} = \text{const}$, from which it follows that the integral curves on the v, v' plane are symmetric relative to the v-axis (Fig. 2).

Hence, the shape of the string should be symmetric with respect to the axis passing through its center; the maximum displacement v_m is achieved at the center. The constant in the first integral equals $(1 - v_m)^{-1}$. Further integration taking account of the condition $v(0) = 0$ yields

$$\tau = \sqrt{{}^1/_2(1 - v_m)} \int_0^v \left(\frac{1 - z}{v_m - z}\right)^{1/2} dz =$$

$$= \sqrt{{}^1/_2 v_m (1 - v_m)} - \sqrt{{}^1/_2(1 - v_m)(1 - v)(v_m - v)} + \tag{3.1}$$

$$+ \frac{\sqrt{2}}{4}(1 - v_m)^{3/2} \ln \frac{(1 + \sqrt{v_m})(\sqrt{1 - v} - \sqrt{v_m - v})}{(1 - \sqrt{v_m})(\sqrt{1 - v} - \sqrt{v_m - v})} \qquad 0 \leqslant v \leqslant v_m,\ 0 \leqslant \tau \leqslant {}^1/_2 \varkappa l$$

For $\varkappa l/2 \leqslant \tau \leqslant \varkappa l$ the dependence $v(\tau)$ is determined by the equality $v(\tau) = v(\varkappa l - \tau)$. The shape of the string is now found to the accuracy of the constant v_m. To determine it, its equivalent relationship $v(\varkappa l/2) = v_m$ is needed. Setting $\tau = \varkappa l/2$, $v = v_m$ in (3.1), we obtain

$$\varkappa l = F(v_m) \equiv \sqrt{2 v_m (1 - v_m)} + + \frac{\sqrt{2}}{2}(1 - v_m)^{3/2} \ln \frac{1 + \sqrt{v_m}}{1 - \sqrt{v_m}} \tag{3.2}$$

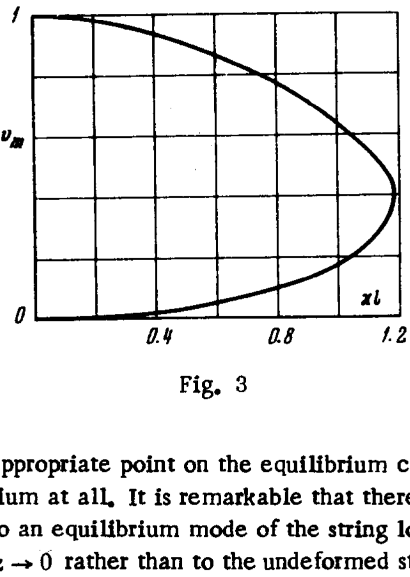

Fig. 3

The dependence of the maximum deflection v_m on the single dimensionless parameter $\varkappa l$ is thereby determined. Constructing the curve $v_m = v_m(\varkappa l)$ (Fig. 3), called the equilibrium curve, we find that for one value of the parameter the string can have either two equilibrium modes, or one mode (the appropriate point on the equilibrium curve is called the limit point), or have no equilibrium at all. It is remarkable that there exists a series of modes (upper branch) tending to an equilibrium mode of the string loaded by a concentrated force at the center as $\varkappa \to 0$ rather than to the undeformed state.

Let us investigate the stability of the equilibrium. Let us proceed as in [1], Ch. VIII, i.e. despite the fact that the number of degrees of freedom is infinite, we make the following two assumptions. We will consider the equilibrium mode stable if it communicates a minimum in the class of functions $v(\tau)$, $0 \leqslant \tau \leqslant \varkappa l$ to the functional

$$V = \Pi - W_0 = \int_0^{\varkappa l} \left(\frac{1}{2} v'^2 - \frac{1}{1 - v}\right) d\tau \tag{3.3}$$

such that $v(0) = v(\varkappa l) = 0$ and $v' \in L_2$. Let us also take the Poincaré deduction on the shift of stability on the equilibrium curve, expounded in [1], Ch. VIII, p. 102, say. Let us find the second variation of V

$$\delta^2 V = \int_0^{\varkappa l} \left[\frac{1}{2} \zeta'^2 - \frac{1}{(1 - v)^3} \zeta^2\right] d\tau \tag{3.4}$$

Here ζ is a function from the mentioned class. We also have

$$\delta^2 V > \int_0^{\varkappa l} \left[\frac{1}{2}\zeta'^2 - \frac{1}{(1-v_m)^3}\zeta^2\right] d\tau \tag{3.5}$$

for all ζ. Let us examine the equilibrium of the lower branch, where $v_m = 0$ for $\varkappa l = 0$. Hence, there exists a $(\varkappa l)_*$ such that for $0 < \varkappa l < (\varkappa l)_*$ the following inequality is valid:

$$\frac{1}{[1 - v_m(\varkappa l)]^3} < \frac{\pi^2}{2(\varkappa l)^2} \tag{3.6}$$

For the same values of the parameter $\varkappa l$

$$\delta^2 V > \frac{1}{2}\int_0^{\varkappa l} \zeta'^2 d\tau - \frac{1}{2}\frac{\pi^2}{(\varkappa l)^2}\int_0^{\varkappa l} \zeta^2 d\tau \tag{3.7}$$

But it is known (see [2], Ch. VII, p. 257, for example) that the right side of (3.7) is nonnegative for all ζ from the given class. Hence, $\delta^2 V > 0$ for $0 \leqslant \varkappa l < (\varkappa l)_*$; for these values of $\varkappa l$ the equilibria of the lower branch are stable. Therefore, the whole lower branch is stable. Stability vanishes at the limit point; the upper branch is unstable.

The same deductions on stability are obtained if the stability of the equilibria adjoining the undeformed state is investigated by seeking the frequencies of small oscillations around these equilibria in the form of power series in $\varkappa$. Physically, the stability of equilibrium sufficiently close to the undeformed state is evident.

Now, let us mention two cases when the imperfection of the ferromagnet can be taken into account. Let a string and magnet be connected, according to Fig. 1, by a magnetic circuit of length $l_1 \gg l$ with the permeability μ_1. Under the assumptions of Sect. 2, along the string $\varphi_2 - \varphi_1 = \varphi = \text{const}$, but $\varphi \neq I$; in the magnetic circuit $B = B_1 = \text{const}$. We have $\varphi + B_1 l_1/\mu = I$. The second relationship between φ and B_1 is obtained from the condition that the magnetic flux Φ through the "lower" surface of the string equals the flux in the magnetic circuit. From (2.3) we find

$$\Phi = \mu_0 \varphi b \int_0^l \frac{dx}{\Delta_0 - u(x, \varphi)} = B_1 S_1 \tag{3.8}$$

Here S_1 is the cross section of the magnetic circuit, and $u(x, \varphi)$ is determined from (3.1) and (3.2), where it is necessary to set $\varkappa = \mu_0 b \varphi^2 / 2T_1\Delta_0^3$. From the two equations in φ and B_1 obtained, they can generally be found. This case corresponds to the estimate $\mu_0/\mu_1 = O(S_1\Delta_0/l_1 lb)$; if this ratio is an order less, then φ and u can be taken at $\mu_1 = \infty$ and B_1 can be found from (3.8). If, however, μ_0/μ_1 is an order higher, φ is found from (3.8) where it is necessary to set $B_1 = I\mu_1/l_1$, and will be small compared with I.

In the second case, the field in the string itself is taken into account but not in the rest of the ferromagnet. Then $\varphi_2 - \varphi_1 = \varphi(x)$. According to Sect. 2, we consider $B = B_2(x)$ in the string.

Let S_2 be the cross section of the string. We have (Fig. 1)

$$\varphi(x) + \frac{1}{\mu_1}\int_x^l B_2(y)\, dy = I \tag{3.9}$$

$$B_2(x) = \frac{1}{S_2}\Phi(x) = \frac{b\mu_0}{S_2}\int_0^x \frac{\varphi(y)}{\Delta_0 - u(y)} dy \tag{3.10}$$

Here $\Phi(x)$ is the magnetic flux through a part of the lower surface of the string included between the left end and the section with coordinate x. By substituting (3.10) into (3.9) and adding the equation

$$\frac{d^2u}{dx^2} + \frac{1}{2}\,\frac{\mu_0\varphi'^2(x)}{[\Delta_0 - u(x)]^2} = 0 \tag{3.11}$$

we arrive at a system consisting of two equations, an integral one and a differential one, for the scalar potential $\varphi(x)$ and the displacement $u(x)$. Such a consideration corresponds to the estimate $\mu_0/\mu_1 = O(\Delta_0 S_2/bl^2)$. If this ratio is an order less, then φ and u can be considered equal to their values for $\mu_1 = \infty$, and B_2 is determined from (3.10). If, however, μ_0/μ_1 is an order greater, then almost all the lines of force of the field will enter the string at its right end $x = l$ and the derivative $\varphi'(x)$ will be quite large in this domain, which contradicts the assumptions of Sect. 2.

4. Equilibrium of a ferromagnetic membrane. Emden-Fowler equation with a negative power of the unknown in the nonlinear member. Let us examine the axisymmetric equilibrium of a circular membrane. In place of (2.5) we obtain the equation

$$\frac{d^2v}{d\rho^2} + \frac{1}{\rho}\,\frac{dv}{d\rho} + \frac{1}{(1-v)^2} = 0 \tag{4.1}$$

with the boundary conditions $v(\varkappa R) = 0$, $-\infty < v(0) < 1$. Here $v = v(\rho)$, $\rho = \varkappa r$ is a dimensionless radial coordinate, and R the membrane radius. After substituting $w = 1 - v$ we arrive at a variety of the Emden-Fowler equation

$$\frac{d^2w}{d\rho^2} + \frac{1}{\rho}\,\frac{dw}{d\rho} - w^{-2} = 0 \tag{4.2}$$

The Emden-Fowler equation (see [3], Ch. VII, for example) has been studied only in cases when the power of the unknown in the nonlinear term is positive. Hence, a special investigation is required here. By substituting

$$\rho = \varkappa R e^{-3\tau/2}, \qquad w = ({}^9/_4\,\varkappa^2R^2)^{1/3}\,\eta e^{-\tau} \tag{4.3}$$

Eq. (4.2) is reduced to an "autonomous" second order system

$$\eta^\cdot = \vartheta, \qquad \vartheta^\cdot = 2\vartheta - \eta + \eta^{-2} \qquad (\eta^\cdot = d\eta/d\tau,\ \vartheta^\cdot = d\vartheta/d\tau) \tag{4.4}$$

for which it is necessary to find a solution such that $\eta(0) = ({}^9/_4\varkappa^2R^2)^{-1/3}$ and $\lim\,[\eta(\tau)\,e^{-\tau}]$ is bounded and positive as $\tau \to \infty$. Let us elucidate how the solution of the system (4.4) behaves as $\tau \to \infty$; the behavior of the solution of (4.2) as $\rho \to 0$ is thereby studied (this is a fundamental problem in the theory of the Emden-Fowler equation).

Let us examine phase trajectories of the system (4.4) in that half of the $O\eta\vartheta$ phase plane where $\eta > 0$. The system (4.4) has one singularity $\vartheta = 0$, $\eta = 1$. Setting $\eta - 1 = \zeta$ and linearizing (4.4) near the singularity

$$\zeta^\cdot = \vartheta, \qquad \vartheta^\cdot = 2\vartheta - 3\zeta + \ldots \tag{4.5}$$

we find this point to be an unstable focus (Fig. 4).

The curve $2\vartheta - \eta + \eta^{-2} = 0$ and on $O\eta$ axis separate the half-plane under consideration into four domains. In 1 and 3 the derivative

$$\frac{d\vartheta}{d\eta} = \frac{2\vartheta - \eta + \eta^{-2}}{\vartheta} \tag{4.6}$$

is positive, and ϑ increases on the integral curves in these domains, as η grows; while ϑ decreases in domains 2 and 4 as η grows. Let us find the second derivative

$$d^2\vartheta/d\eta^2 = -\vartheta^{-3}[(1+2\eta^{-3})\vartheta^2 - 2(\eta-\eta^{-2})\vartheta + (\eta-\eta^{-2})^2] \tag{4.7}$$

The quadratic trinominal in ϑ in the square brackets in (4.7) has no real roots for $\eta > 0$, and therefore, does not change sign. Hence, for $\vartheta > 0$ the integral curves are convex upward, and for $\vartheta < 0$, downward. The $O\eta$ axis intersects the line $\vartheta(\eta)$ from the bottom up for $\eta < 1$ and from the top down for $\eta > 1$, and has a vertical tangent. For $\vartheta > 0$ the following inequality is valid:

$$d\vartheta(\eta)/d\eta > d\vartheta_*(\eta)/d\eta \tag{4.8}$$

where $\vartheta_*(\eta)$ is determined by the linear system

$$\eta^\cdot = \vartheta_*, \qquad \vartheta^\cdot_* = 2\vartheta_* - \eta \tag{4.9}$$

It hence follows that the integral curve $\vartheta(\eta)$ of the system (4.4), starting at some point $\eta_0, \vartheta_0 > 0$ will lie, for $\tau > 0$, above the integral curve of the system (4.9) starting from the same point, at least until $\vartheta > 0$. But the line $\vartheta = \eta$ is an integral curve of the system (4.9). Hence, integral curves of the system (4.4) intersect it from the bottom upward. The above permits pointing out the direction of the integral curves in various parts of $\eta > 0$ phase half-plane (arrows in Fig. 4).

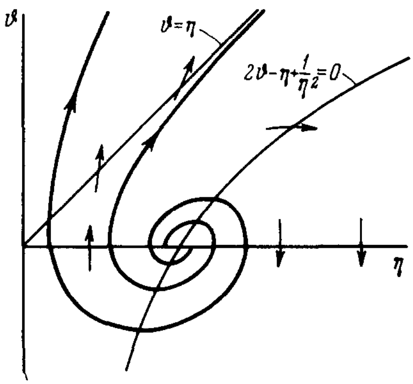

Fig. 4

Now, let us examine the domain $1 < \eta < \infty$, $0 < \vartheta < \eta$ and a segment of some line $\eta = \eta_* =$ const therein, on which $0 \leqslant \vartheta \leqslant \leqslant \eta_*$. Some half-interval $[0, \alpha)$ of this segment is composed of points of integral curves which intersect the $O\eta$ axis from the top down as τ increases further. The other half-interval $(\beta, \eta_*]$ is filled with points of curves still intersecting the line $\vartheta = \eta$ The points α and β cannot belong to the mentioned set since otherwise a "last" trajectory intersecting the $O\eta$ axis or the line $\vartheta = \eta$ would be found. Hence, there exists a closed set of integral curves going to infinity between the lines $\vartheta = \eta$ and $\vartheta = 0$; these curves are henceforth designated as "separating". The half-branch of the separating curve going to infinity starting at some point should lie above the line $2\vartheta - \eta + \eta^{-2} = 0$, hence, η and ϑ grow monotonically on this half-branch. Let us show that the separating curves have the straight line $\vartheta = \eta$ as their asymptote. Integrating the linear part of the system (4.4), we obtain

$$\eta(\tau) = \eta_0 e^\tau + (\vartheta_0 - \eta_0)\tau e^\tau + \int_0^\tau (\tau - \sigma) e^{\tau-\sigma} \eta^{-2}(\sigma)\, d\sigma$$

$$\vartheta(\tau) = \vartheta_0 e^\tau + (\vartheta_0 - \eta_0)\tau e^\tau + \int_0^\tau (1 + \tau - \sigma) e^{\tau-\sigma} \eta^{-2}(\sigma)\, d\sigma \tag{4.10}$$

$$\vartheta(\tau) - \eta(\tau) = (\vartheta_0 - \eta_0) e^\tau + \int_0^\tau e^{\tau-\sigma} \eta^{-2}(\sigma)\, d\sigma \tag{4.11}$$

where η_0, ϑ_0 are the initial data. Let us take the point (η_0, ϑ_0) on the part of the separating curve going to infinity, where η and ϑ are monotonic. Then $\eta(\tau) > \eta_0$ for $\tau > 0$, and

$$\vartheta(\tau) < \vartheta_0 e^{\tau} + (\vartheta_0 - \eta_0)\,\tau e^{\tau} + \int_0^{\tau} (1 + \tau - \sigma)\, e^{\tau-\sigma} \eta_0^{-2} d\sigma = \vartheta_0 e^{\tau} + (\vartheta_0 - \eta_0 + \eta_0^{-2})\,\tau e^{\tau}$$

If the coefficient of τe^{τ} in the last part of the relationship (4.11) is negative, $\vartheta(\tau)$ starts to decrease for some value of τ. Since this is impossible on the part of the separating curve under consideration, there is no point thereon where $\vartheta - \eta + \eta^{-2} < 0$. Hence and from the condition $\eta - \vartheta > 0$ it follows that $\eta - \vartheta \to 0$ as $\tau \to \infty$. Therefore the line $\vartheta = \eta$ is an asymptote for any separating curve. There is also obtained from (4.6) that $d\vartheta/d\eta \to 1$ as $\tau \to \infty$.

Furthermore, let us show that there exists only one separating curve. Let us assume the opposite, and let us examine the "monotone" half-branches of two separating curves $\vartheta_1(\eta)$ and $\vartheta_2(\eta)$ which go to infinity. Let $\vartheta_1(\eta_0) > \vartheta_2(\eta_0)$, $\eta_0 > 1$. The functions $\vartheta_1(\eta)$, $\vartheta_2(\eta)$ exist for all $\eta > \eta_0$ on the considered portions of the integral curves, are positive and increase monotonically, tending asymptotically to the line $\vartheta = \eta$. Let us use the notation $\Delta\vartheta = \vartheta_1 - \vartheta_2$. From (4.6) we have

$$\frac{d}{d\eta}\,\Delta\vartheta = (\eta - \eta^{-2})\,\frac{\Delta\vartheta}{\vartheta_2(\vartheta_2 + \Delta\vartheta)} \tag{4.12}$$

Since the curves $\vartheta_1(\eta)$ and $\vartheta_2(\eta)$ have the same asymptote, Eq. (4.12) should admit a solution $\Delta\vartheta(\eta)$ for $\Delta\vartheta(\eta_0) = \vartheta_1(\eta_0) - \vartheta_2(\eta_0) > 0$ such that $\Delta\vartheta$ exists and is positive for all $\eta > \eta_0$ and $\Delta\vartheta > 0$ as $\eta \to \infty$. But (4.12) has no such solution. Indeed, if $\Delta\vartheta > 0$, then $d(\Delta\vartheta)/d\eta > 0$ also, and a positive solution of (4.12) cannot decrease. Therefore, if $\vartheta(\eta)$ is a separating curve, the integral curve passing above it for sufficiently large η cannot be separating curve. Therefore, even two separating curves don't exist.

The behavior of the integral curves on the whole $\eta > 0$ half-plane can now be described. Any curve, except the singular $\eta \equiv 1$ and the separating curve making an infinite number of turns around the focus, intersects the line $\vartheta = \eta$ and goes to infinity above this line "parallel" to it in the sense that $\lim (d\vartheta/d\eta) = 1$ as $\eta \to \infty$. The separating curve going out of the focus will approach the line $\vartheta = \eta$ asymptotically from below (Fig. 4).

On the basis of the geometric properties established for the integral curves, let us elucidate how rapidly the functions $\eta(\tau)$ and $\vartheta(\tau)$ increase as $\tau \to \infty$. The answer to this question is given by the following theorem: as $\tau \to \infty$ the $\lim [\eta(\tau)\,\tau^{-1} e^{-\tau}]$ is finite and positive on all the integral curves in the $\eta > 0$ half-plane except the singular solution $\eta \equiv 1$ and the separating curve; a finite and positive $\lim [\eta(\tau)\, e^{-\tau}]$ exists on the separating curve; the properties of $\vartheta(\tau)$ are the same. Let us examine two solutions $\eta_1(\tau)$ and $\eta_2(\tau)$ of the system (4.4) with the initial conditions $\eta_{1,2}(0) = \eta_{10,20}$, $\vartheta_{1,2}(0) = \vartheta_{10,20}$ such that the points $(\eta_{10}, \vartheta_{10})$ and $(\eta_{20}, \vartheta_{20})$ lie on one integral curve. Let the passage from the former to the latter be performed in an interval τ_{12}, i.e. $\eta_1(\tau_{12}) = \eta_{20}$, $\vartheta_1(\tau_{12}) = \vartheta_{20}$; $\tau_{12} > 0$ if this passage corresponds to the rise in τ and $\tau_{12} < 0$ if it corresponds to a decrease. We have

$$\eta_2(\tau) = \eta_1(\tau + \tau_{12}), \quad \lim_{\tau\to\infty} [f(\tau)\,\eta_1(\tau)] = \left[\lim_{\tau\to\infty} \frac{f(\tau + \tau_{12})}{f(\tau)}\right]\left[\lim_{\tau\to\infty} f(\tau)\,\eta_2(\tau)\right] \tag{4.13}$$

under the conditions that the mentioned limits exist. Let us take an integral curve distinct from the separating curve, and let us select the origin of reference τ at a point where $\vartheta > \eta$. Since $\eta(\tau)$ increases monotonically for $\tau > 0$, on the basis of (4.10) we can write

$$\vartheta_0 - \eta_0 < \eta(\tau)\,\tau^{-1}e^{-\tau} < \vartheta_0 - \eta_0 + \eta_0\tau^{-1} + \eta_0^{-2}\,[1 - \tau^{-1} + \tau^{-1}e^{-\tau}] \qquad (4.14)$$

The first of these inequalities is evident, the second is obtained analogously to (4.11). It hence follows that the function $\eta(\tau)\,\tau^{-1}e^{(-\tau)}$ has an upper and lower bound as $\tau \to \infty$ Let us evaluate the derivative

$$[e^{-\tau}\tau^{-1}\eta(\tau)]^{\cdot} = \tau^{-2}\Big[\int_0 \sigma e^{-\sigma}\eta^{\,2}(\sigma)\,d\sigma - \eta_0\Big] \qquad (4.15)$$

The function in the square brackets in the right side of (4.15) is monotonically increasing, and therefore, does not change sign for sufficiently large τ. Hence, the function $\eta(\tau)\,\tau^{-1}e^{-\tau}$ is monotone at infinity, and as a function monotone and bounded on both sides must have a limit as $\tau \to \infty$. For different η_0, ϑ_0 the values of this limit are connected by means of the relationship

$$\lim\,[\eta_1(\tau)\,\tau^{-1}e^{-\tau}] = \exp(-\tau_{12})\lim\,[\eta_2(\tau)\,\tau^{-1}e^{-\tau}]$$

It is hence seen that the values of the limit are always positive. From the relationship

$$\vartheta(\tau) - \eta(\tau) < (\vartheta_0 - \eta_0 + \eta_0^2)\,e^{\tau} - \eta_0^{-2}, \qquad \vartheta(\tau) - \eta(\tau) > 0 \quad \text{for} \quad \tau \to \infty \qquad (4.16)$$

there results that

$$\lim\,[(\vartheta - \eta)\,\tau^{-1}e^{-\tau}] = 0, \qquad \lim\,[\vartheta(\tau)\,\tau^{-1}e^{-\tau}]$$

exists and

$$\lim\,[\vartheta(\tau)\,\tau^{-1}e^{-\tau}] = \lim\,[\eta(\tau)\,\tau^{-1}e^{-\tau}]$$

Now, let us consider the separating curve. Let us use the identity

$$\eta(\tau)\,e^{-\tau} = \eta_0\exp\int_0^{\tau}[\eta^{\cdot}(\sigma) - \eta(\sigma)]\,\frac{d\sigma}{\eta(\sigma)} \qquad (4.17)$$

It hence follows that if $\lim\,[\eta(\tau)\,e^{-\tau}]$ exists, the integral in the right side of the relationship

$$\lim_{\tau\to\infty}\,[\eta(\tau)\,e^{-\tau}] = \eta_0\exp\int_0^{\infty}[\vartheta(\sigma) - \eta(\sigma)]\,\frac{d\sigma}{\eta(\sigma)} \qquad (4.18)$$

converges, and conversely, if the integral converges, the limit exists. Let us show convergence of the integral. Differentiating (4.4), we obtain for $\eta > 1$

$$\eta^{\cdot\cdot\cdot} = \vartheta^{\cdot\cdot} = \vartheta - 2(\eta - \vartheta)(1 - \eta^{-3}) > \vartheta - 2(\eta - \vartheta) \qquad (4.19)$$

Let us select the origin of reference τ so that

$$\eta_0 > 1, \quad \vartheta^{\cdot}(0) > 0, \quad \vartheta_0 - 2(\eta_0 - \vartheta_0) > 0$$

and the functions η, ϑ would grow monotonically as $\tau > 0$; as is seen from the preceding, such a choice is possible on the separating curve. The inequality (4.19) then yields $\vartheta^{\cdot\cdot} > 0$ for $\tau > 0$. Further, we find

$$\vartheta^{\cdot}(\tau) = \int_0^{\tau}\vartheta^{\cdot\cdot}(\sigma)\,d\sigma + \vartheta^{\cdot}(0) > \vartheta^{\cdot}(0)$$

$$\vartheta(\tau) = \int_0^{\tau}\vartheta^{\cdot}(\sigma)\,d\sigma + \vartheta_0 > \vartheta^{\cdot}(0)\,\tau + \vartheta_0$$

$$\eta(\tau) = \int_0^{\tau}\vartheta(\sigma)\,d\sigma + \eta_0 > {}^1/_2\vartheta^{\cdot}(0)\,\tau^2 + \vartheta_0\tau + \eta_0 \qquad (4.20)$$

This latter inequality shows that the integral of $\eta^{-1}(\tau)$ converges in $[0, \infty)$. But since $(\eta - \eta^{\cdot}) = (\eta - \vartheta) \to 0$ as $\tau \to \infty$, integral in (4.18) also converges. Therefore, on the separating curve there exists

$$\lim\,[\eta(\tau)\,e^{-\tau}] \quad \text{and}\ 0 < \lim\,[\eta(\tau)\,e^{-\tau}] < \eta_0$$

Evidently

$$\lim\,[\vartheta(\tau)\,e^{-\tau}] = \lim\,[\eta(\tau)\,e^{-\tau}]$$

Values of the limits for different initial conditions are connected by the relationships

$$\lim\,[\eta_1(\tau)\,e^{-\tau}] = \exp(-\tau_{12})\lim\,[\eta_2(\tau)e^{-\tau}],\ \eta_1(\tau_{12}) = \eta_2(0)$$

Let us turn to the initial boundary value problem. As has been mentioned, one of the boundary conditions requires that $0 < \lim\,[\eta(\tau)\,e^{-\tau}] < \infty$. Hence, the desired function $\eta(\tau)$ must be such that $\eta(\tau)$, $\vartheta(\tau)$ would yield a parameteric representation of the separating curve. The measurement of τ on this curve is given by the second condition $\eta(0) = \eta_0 = (^9/_4\,\varkappa^2R^2)^{-1/3}$. Therefore, for each point of intersection of the line $\eta = \text{const} = (^9/_4\,\varkappa^2R^2)^{-1/3}$ with the separating curve there is a function $\eta(\tau)$ with the requisite properties, and the values of $\eta(\tau)$ for $0 \leqslant \tau < \infty$ determine the equilibrium mode according to (4.3).

The functions $\eta(\tau)$ and $\vartheta(\tau)$ utilized later correspond to the separating curve. Evaluating the derivative

$$dw\,/\,d\rho = (^3/_2\varkappa R)^{-1/3}(\eta - \vartheta)\,e^{\tau/2} > 0 \tag{4.21}$$

by using (4.3), we find that w grows, and the deflection v decreases as ρ increases from zero to $\varkappa R$. It has been established earlier that $\eta - \vartheta < \eta^{-2}$ on the monotone half-branch of the separating curve; hence $0 < (\eta - \vartheta)\,e^{2\tau} < [\eta e^{-\tau}]^{-2}$. It is thereby shown that the function $(\eta - \vartheta)\,e^{2\tau}$ is bounded at infinity, and therefore, $(\eta - \vartheta)\,e^{\tau/2} \to 0$ as $\tau \to \infty$ or $\rho \to 0$, i. e. at the center the membrane has a tangent plane parallel to the plane of the contour.

Let v_m denote the deflection at the center. From (4.3) we have

$$v_m = 1 - w_m = 1 - \eta_0^{-1}\lim_{\tau\to\infty}\,[\eta(\tau)\,e^{-\tau}] \tag{4.22}$$

Let us construct the equilibrium curve $v_m = v_m(\varkappa R)$. The number of solutions is hence also determined. Let us examine equilibria close to the undeformable state. We have

$$\lim_{\eta_0\to\infty}\{\eta_0^{-1}\lim_{\tau\to\infty}\,[\eta(\tau)\,e^{-\tau}]\} =$$

$$= \lim_{\tau_{10}\to\infty}\left\{\exp\int_0^\infty[\eta_1'(\tau+\tau_{10}) - \eta_1(\tau+\tau_{10})]\,\eta_1^{-1}(\tau+\tau_{10})\,d\tau\right\} = 1 \tag{4.23}$$

where $\eta_1(\tau)$ is a function with any fixed initial value, and $\eta_1(\tau_{10}) = \eta_0$. Hence $v_m \to 0$ as $\eta_0 \to \infty$ or $\varkappa R \to 0$, as should be for equilibria of this series.

Let η_{*i}, $i = 1, 2, \ldots$ denote points of intersection of the separating curve with the $O\eta$ axis; their numbering corresponds to motion along the curve as τ decreases. Let us consider two functions $\eta_1(\tau)$ and $\eta_2(\tau)$ with initial values on that half-branch of the separating curve where η and ϑ are monotone.

Let

$$\eta_2(0) = \eta_{20} > \eta_1(0) = \eta_{10}, \qquad \eta_1(\tau_{12}) = \eta_{20}$$

Two values of the parameter $(\varkappa R)_2 < (\varkappa R)_1$ correspond to solutions of $\eta_1(\tau)$, $\eta_2(\tau)$. By going from η_{10} to η_{20} along the monotone half-branch where $0 < \vartheta < \eta$, we obtain

$$\tau_{12} = \int_{\eta_{10}}^{\eta_{20}}\frac{d\eta}{\vartheta(\eta)} > \ln\frac{\eta_{20}}{\eta_{10}} \tag{4.24}$$

Hence, and from (4.13)

$$w_{m1} = \eta_{10}^{-1}\lim_{\tau\to\infty}\,[\eta_1(\tau)\,e^{-\tau}] = \eta_{20}\eta_{10}^{-1}w_{m2}\exp(-\tau_{12}) < w_{m2}, \quad v_{m1} > v_{m2} \tag{4.25}$$

It follows from the form of the separating curve that for sufficiently small $\varkappa R$ (or large η_0) it has one and only one point of intersection with the line $\eta = \eta_0$. Hence, for small values of the parameter there exists a unique equilibrium mode (in contrast to the string, Fig. 3). An equilibrium series which exist for

$$0 \leqslant \varkappa R \leqslant (\varkappa R)_{*1} = {}^2/_3 \eta_{*1}^{-3/2}$$

is started by the set of these modes which adjoin the undeformed state according to (4.23).

The inequalities (4.25) show that the maximum deflection v_m for equilibria of this series increases monotonically from zero to some v_{m*1} as $\varkappa R$ increases continuously from zero to $(\varkappa R)_{*1}$. Let us evaluate the derivative

$$\frac{dv_m}{d(\varkappa R)} = \frac{d\eta_0}{d(\varkappa R)} \frac{dv_m}{d\eta_0} = -\eta_0^{5/2}\left[-\frac{d}{d\eta_0}\exp\int\limits_0^\infty \frac{\vartheta(\tau)-\eta(\tau)}{\eta(\tau)}\,d\tau\right] =$$

$$= \eta_0^{5/2}\left[\frac{d}{d\tau_{10}}\exp\int\limits_{\tau_{10}}^\infty \frac{\vartheta_1(\tau)-\eta_1(\tau)}{\eta_1(\tau)}\,d\tau\right]\frac{d\tau_{10}}{d\eta_0} = (1-v_m)\,\eta_0^{3/2}\,\frac{\eta_0-\vartheta_0}{\vartheta_0} \qquad (4.26)$$

Here $\eta_1(\tau)$ and τ_{10} are introduced as in (4.23). From the inequality $\eta - \vartheta < \eta^{-2}$ there follows $dv_m/d(\varkappa R) = 0$ for $\varkappa R = 0$. For $\varkappa R = 0$ the equilibrium curve has a horizontal, and for $\varkappa R = (\varkappa R)_{*1}$ a vertical tangent. Equilibrium is impossible for $\varkappa R > (\varkappa R)_{*1}$.

Equilibria of the mentioned series are unique only for $\varkappa R < (\varkappa R)_{*2}$. For $(\varkappa R)_{*2} < \varkappa R < (\varkappa R)_{*1}$ the line $\eta = \eta_0(\varkappa R)$, in addition to the monotone half-branch, intersects the lower curl of the separating curve connecting the points η_{*1} and η_{*2}. This yields a new segment of the equilibrium curve. Let us consider two functions $\eta_1(\tau)$ and $\eta_2(\tau)$ with the initial values $\eta_{20} > \eta_{10}$ on the lower curl. Passage from the point with $\eta = \eta_{10}$ to the point with $\eta = \eta_{20}$ now corresponds to a decrease in τ, and if $\eta_1(\tau_{12}) = \eta_{20}$, then $\tau_{12} < 0$. Hence

$$w_{m1} = \eta_{20}\eta_{10}^{-1} w_{m2} \exp(-\tau_{12}) > w_{m2}, \qquad v_{m1} < v_{m2}$$

i.e. the maximum deflection for equilibria of this series decreases as the parameter increases. Here, in particular, any $v_m > v_{m*1}$. At the extreme points of this segment of the equilibrium curve $(\varkappa R)_{*2}$, $(\varkappa R)_{*1}$ the tangents are vertical. Since the dependence $v_m(\varkappa R)$ is continuous, then for $(\varkappa R)_{*1}$ the two considered branches of the equilibrium curve join, and $(\varkappa R)_{*1}$ corresponds to the limit point.

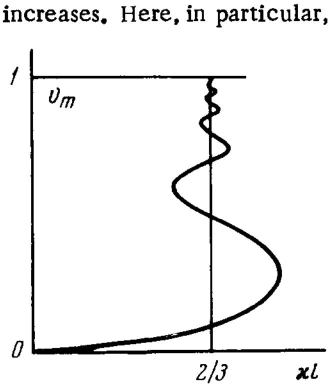

Fig. 5

By the same means it is shown that still another ascending branch of the equilibrium curve lies between $(\varkappa R)_{*2}$, $(\varkappa R)_{*3}$, i.e. where v_m increases as $\varkappa R$ grows, a descending branch lies between $(\varkappa R)_{*4}, \varkappa R)_{*5}$, etc., and all the values of $(\varkappa R)_{*i}$ correspond to limit points. We hence obtain an equilibrium curve of a rare kind, shown in Fig. 5, where $\varkappa l = \varkappa R$.

The equilibrium curve as an infinite number of branches corresponding to the infinite number of curls of the separating curve, and intersects the line $\varkappa R = {}^2/_3$, an infinite number of times, approaching the point $\varkappa R = {}^2/_3$, $v_m = 1$. This point correponds to the singular solution $\eta \equiv 1$ or $v(\rho) = 1 - (3\rho/2)^{2/3}$; it can be found at once from (4.1) if a solution of the form $v = 1 - C\rho^\alpha$ is sought, where C, $\alpha = \mathrm{const}$. The number of solutions which the boundary value problem considered admits for the given value of the parameter is the following: if $\varkappa R > (\varkappa R)_{*1}$, there are no solutions; if $\varkappa R < (\varkappa R)_{*2}$ and also for $\varkappa R = (\varkappa R)_{*1}$ there is one solution; if $(\varkappa R)_{*3} < \varkappa R < (\varkappa R)_{*1}$ or $\varkappa R = (\varkappa R)_{*2}$ there are two solutions, etc.; there are ranges of variation of $\varkappa R$ where exactly n solutions exist (n is an integer); for $\varkappa R = {}^2/_3$ there is a countable set of solutions.

It is easy to show that the equilibria close to the undeformed state are stable. As in Sect. 3, we hence deduce that all the ascending branches of the equilibrium curve correspond to stable modes, and the descending branches to unstable modes. However, this deduction is valid only under the condition that nonaxisymmetric equilibrium modes either do not exist, or their branch does not intersect the branch of axisymmetric modes. These facts have not been proved. Attention is turned to the essential distinctions between the two- and one-dimensional cases, the membrane and the string. They are apparently connected with the fact that the membrane, in contrast to the string, cannot bear concentrated forces.

5. Equilibrium of elastic conductors. Let us consider the equilibrium of two thin conductors over which current flows according to Sect. 1. Let us consider the conductors to interact on a section of considerable length, where the spacings between them are commensurate with the displacements, and small in comparison to the length and radii of curvature of the current lines. The conductors will then be "approximately" parallel on the mentioned section. The cross-sectional dimensions are assumed to be of the same order as the displacement (for a rod, for example), or less (as for a string). Let us assume $\mu = \mu_0$ in the whole space.

We shall first examine an auxiliary problem. Let be given a linear conductor and a point M between which the spacing $r_0 = |\mathbf{r}_0|$ is small. The field intensity at M is

$$\mathbf{H}(M) = \frac{I}{4\pi} \oint \frac{\boldsymbol{\tau} \times \mathbf{r}}{r^3}\, ds \tag{5.1}$$

where I is the current, $\boldsymbol{\tau}$ is the unit vector of the tangent in the current direction, $\mathbf{r}$ is a vector connecting M with a point on the conductor. Let us measure s so that

$$\mathbf{r}(s) = \mathbf{r}_0, \qquad \boldsymbol{\tau}(s) = \boldsymbol{\tau}_0 \qquad \text{for } s = 0$$

$$-l_1 \leqslant s \leqslant l - l_1, \qquad \mathbf{r}(-l_1) = \mathbf{r}(l - l_1)$$

We have

$$\mathbf{H}(M) = \frac{I}{4\pi} \int_{-l_1}^{l-l_1} \frac{\boldsymbol{\tau}_0 \times \mathbf{r}_0 + \ldots}{(r_0^2 + s^2 + \ldots)^{3/2}}\, ds = \frac{I}{2\pi r_0}\, \mathbf{b}_0 + \delta\mathbf{H}, \quad \mathbf{b}_0 = \frac{\boldsymbol{\tau}_0 \times \mathbf{r}_0}{r_0} \tag{5.2}$$

The addition $\delta\mathbf{H}$ corresponding to the members not written down in the integrand is such that $r_0\,|\delta\mathbf{H}| \to 0$, as $r_0 \to 0$. Hence, we shall discard it in such cases, i.e. we evaluate $\mathbf{H}$ at points near the conductors by replacing the closed curvilinear conductor by an infinite straight conductor directed along $\boldsymbol{\tau}_0$ and at a distance r_0 away. However, the accuracy of the result will be lower than in Sects. 2, 3 since $|\delta\mathbf{H}|$ generally contains terms of the form $Ik \ln (l/r_0)$, where k is the curvature.

Let us evaluate the load acting on the conductors. Assuming the cross section to vary sufficiently slowly with length, we can ignore the vectors $\mathbf{j}$ being nonparallel in the conductor cross section. In "almost parallel" conductors the plane perpendicular to the vectors $\mathbf{j}$ will be "almost" perpendicular in one conductor, to the vectors $\mathbf{j}$ in the other conductor (a difference of an order in the ratio of the spacings between the conductors and their length). Hence, rotation of the cross sections need not be taken into account in evaluating the load in the deformed state. Let us draw a plane perpendicular to the currents, and let σ_1, σ_2 denote the sections being formed, and M, N points in σ_1 and σ_2.

With the accuracy accepted, let us find the field intensity produced by currents of the second conductor in the first conductor

$$\mathbf{H}(M) = \frac{1}{2\pi} \int_{\sigma_2} \frac{\mathbf{j}(N) \times \mathbf{r}(M, N)}{r^2(M, N)} d\sigma_2, \qquad \mathbf{r}(M, N) = \overrightarrow{MN} \tag{5.3}$$

The interaction between currents flowing in the same conductor also causes some deformations, however, its influence on bending need not be taken into account. The volume forces produced by the field (5.3) in the first conductor are

$$\mathbf{f}(M) = \frac{\mu_0}{2\pi} \int_{\sigma} \frac{\mathbf{j}(M) \times [\mathbf{j}(N) \times \mathbf{r}(M, N)]}{r^2(M, N)} d\sigma_2 = \frac{\gamma\mu_0}{2\pi} \int_{\sigma_2} \frac{j(M)\, j(N)\, \mathbf{r}(M, N)}{r^2(M, N)} d\sigma_2 \tag{5.4}$$

where $\gamma = -1$ if the vectors $\mathbf{j}(M)$ and $\mathbf{j}(N)$ are parallel, and $\gamma = 1$ if they are antiparallel.

Only the linear loading $\mathbf{q}$ determined by integrating $\mathbf{f}(M)$ over the cross section

$$\mathbf{q} = \frac{\gamma\mu_0}{2\pi} \int_{\sigma_1} \int_{\sigma_2} \frac{j(M)\, j(N)\, \mathbf{r}(M, N)}{r^2(M, N)} d\sigma_2 d\sigma_1 \tag{5.5}$$

is essential in the approximate theory of bending.

The forces $\mathbf{f}(M)$ produce distributed bending moments and torques also, but their sum in a segment of a length commesurate to the length of the conductor is on the order of $fr_0^3 l$ while the order of the bending moment due to the loading $\mathbf{q}$ is $fr_0^2 l^2$; the distributed moments need not be taken into account.

For a constant cross section with uniform current distribution the loading is a function of only the displacements in $\mathbf{r}$, but in contrast to (2.9), its form depends on the shape of the cross section.

In order to obtain the equilibrium equation, the appropriate equations of elasticity theory should be written down, and the loading therein should be expressed in terms of the displacements according to (5.5). Let us form these equations for the interaction of two initially parallel strings (Fig. 6), whose cross sections have negligibly small dimensions compared with the displacements and the initial spacing Δ_0. We have from (5.5)

$$\mathbf{u}_1'' + \frac{\gamma\mu_0 I_1 I_2}{2\pi} \frac{\mathbf{u}_2 - \mathbf{u}_1 + \Delta_0 \mathbf{i}_2}{|\mathbf{u}_2 - \mathbf{u}_1 + \Delta_0 \mathbf{i}_2|^2} = 0 \qquad \left(\mathbf{u}_1' = \frac{d\mathbf{u}_1}{dx}\right)$$

$$\mathbf{u}_2'' - \frac{\gamma\mu_0 I_1 I_2}{2\pi} \frac{\mathbf{u}_2 - \mathbf{u}_1 + \Delta_0 \mathbf{i}_2}{|\mathbf{u}_2 - \mathbf{u}_1 + \Delta_0 \mathbf{i}_2|^2} = 0 \qquad \begin{pmatrix} \mathbf{u}_1 = v_1 \mathbf{i}_2 + w_1 \mathbf{i}_3 \\ \mathbf{u}_2 = v_2 \mathbf{i}_2 + w_2 \mathbf{i}_3 \end{pmatrix} \tag{5.6}$$

Here I_1, I_2 are currents, $\mathbf{u}_1$ the displacement of points of the first and second strings, respectively, $\mathbf{i}_2$, $\mathbf{i}_3$ the unit vectors of the axes Oy and Oz, where the vector $\mathbf{i}_2$ is directed downward, and the vector $\mathbf{i}_3$ perpendicular to the plane xy. Combining equations (5.6), we find $\mathbf{u}_1'' + \mathbf{u}_2'' = 0$.

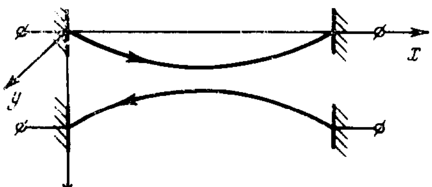

Fig. 6

From the boundary conditions $\mathbf{u}_1(0) = \mathbf{u}_2(0) = 0$, $\mathbf{u}_1(l) = \mathbf{u}_2(l) = 0$ it now follows:

$$\mathbf{u}_1 \equiv -\mathbf{u}_2$$

Let us introduce the notation

$$\varkappa^2 = \frac{\mu_0 I_1 I_2}{2\pi\Delta_0}, \qquad \tau = \varkappa x, \qquad u = \frac{(\Delta_0 - 2v_1)}{\Delta_0}, \qquad w = \frac{2w_1}{\Delta_0}$$

Here u is the dimensionless distance to the plane of symmetry. Eliminating the variable u_2 in (5.6), taking projections on the axes and passing to dimensionless coordinates, we obtain

$$u^{\cdot\cdot} - \gamma \frac{u}{u^2+w^2} = 0, \quad w^{\cdot\cdot} - \gamma \frac{w}{u^2+w^2} = 0 \qquad \left(u^{\cdot} = \frac{du}{d\tau}\right) \tag{5.7}$$

Equations (5.7) agree with the equations of motion of a material point in a central force field whose magnitude is inversely proportional to the distance. Motion occurs in a plane where u, w are Cartesian coordinates, and the center of attraction or repulsion is at the origin. The projection of the string on this plane will be an orbit. It is here required to find the trajectory which starting at the point $u = 1$, $w = 0$ for $\tau = 0$ will again arrive to the same point for $\tau = \varkappa l$.

For an attracting string $\gamma = 1$, to which correspond motions subjected to repulsive forces. Evidently closed or self-intersecting trajectories here do not exist. Hence, a body can return to the initial position only for a motion along a line passing through the center under the condition that its velocity was initially directed towards the center. Only these motions can indeed correspond to solutions of the boundary value problem under consideration. Since the initial point $u = 1$, $w = 0$ lies on the axis Ou passing through the center, then $w \equiv 0$. The equation now obtained for u is integrated and yields

$$\tau = \sqrt{2}\,(1 - v_m) \int\limits_{\varphi_1(v)}^{\varphi_1(0)} \exp z^2 dz, \qquad 0 \leqslant \tau \leqslant \varkappa \frac{l}{2} \tag{5.8}$$

$$\varphi_1(v) = [\ln(1-v) - \ln(1-v_m)]^{1/2}, \qquad v = 2v_1/\Delta_0 = 1 - u$$

Here v is the dimensionless displacement, $v_m < 1$ is the maximum value achieved at the middle of the string. For $0 \leqslant \tau \leqslant \varkappa l/2$ the displacement v increases monotonically from zero to v_m. The relationship (5.8) and the equality $v(\tau) = v(\varkappa l - \tau)$ define the equilibrium mode to the accuracy of the single constant v_m. To determine it an equation analogous to (3.2) is obtained:

$$\varkappa l = 2\sqrt{2}\,(1 - v_m) \int\limits_0^{\psi_1(v_m)} \exp z^2 dz \tag{5.9}$$

where $\psi_1(v_m) = [-\ln(1-v_m)]^{1/2}$. This yields an equilibrium curve similar to that given in Fig. 3; the discussion on stability is also conserved.

The case $\gamma = -1$ corresponding to attraction to the center or repulsion of the string is substantially more complex. Here we find plane equilibrium modes to which motions along a line passing through the center correspond.

For an initial velocity directed from the center, the body will first move from the center, then towards the center, and return to the initial position (the body cannot go to infinity without returning since the inequality $h - \ln u \geqslant 0$ should be satisfied, where h is constant energy). The solution of the original boundary value problem in which $w \equiv 0$ can correspond to such motion (from the beginning of recession to return). Integrating the equation obtained for u, we find

$$\tau = \sqrt{2}\,(1 - v_m) \int\limits_{\varphi_2(v)}^{\varphi_2(0)} \exp(-z^2)\, dz, \qquad 0 \leqslant \tau \leqslant \varkappa \frac{l}{2} \tag{5.10}$$

$$\varkappa l = 2\sqrt{2}\,(1 - v_m) \int\limits_0^{\psi_2(v_m)} \exp(-z^2)\, dz \tag{5.11}$$

$$\varphi_2(v) = [\ln(1 - v_m) - \ln(1 - v)]^{1/2}, \quad \psi_2(v_m) = [\ln(1 - v_m)]^{1/2} \qquad \text{(cont.)}$$

Here v_m is the maximum displacement in absolute value reached at the middle of the string. In this case $v < 0$ for all $\tau \neq 0$, and $v(\tau)$ decreases monotonically from zero to v_m for $0 \leqslant \tau \leqslant \varkappa l/2$. The series of equilibrium modes obtained exists for all $\varkappa l$; the dependence between $\varkappa l$ and v_m for it is mutually single-valued. This series is a continuation of (5.9) in the parameter $\gamma\varkappa l$, however this is insufficient for a judgement of the stability since the stability can vanish upon a series of nonplanar modes branching off from it.

A group of rectilinear motions accompanied by collisions of the center also corresponds to formal solutions of the boundary value problem. Let the initial velocity be directed to the center. In subsequent motion the body will descend to the center, having an infinite velocity here. Considering such motion to be the limit of a sequence of motions in which the body envelops the center on ever more narrow trajectories, we find that the body returns to the initial point after impact with the previous but oppositely directed velocity. A symmetric equilibrium mode with a cusp at the center of the string corresponds to this motion. Here $v > 0$ for $\tau \neq 0$, and $v(\varkappa l/2) = 1$, $v'(\varkappa l/2) = \infty$.

Hence, such solutions contradict the assumption of smallness of the slopes of the string, and can be considered only as formal solutions. At the same time they are of interest for two reasons. Firstly, for a sufficiently long string, equilibrium modes similar to those given are possible everywhere except in a domain near the middle. Secondly, nonplanar modes branch off from these modes, where initial solutions near the bifurcation points must be known in order to investigate their stability.

Integrating (5.7) for $w \equiv 0$ and $u_0' < 0$ we find the connection between $\varkappa l$ (the time of motion) and u_0'

$$\varkappa l = \sqrt{2} \exp(u_0'^2/2) \int\limits_{|u_0'|\sqrt{1/2}}^{\infty} \exp(-z^2)\, dz \tag{5.12}$$

A corresponding sequence of modes exists for $0 \leqslant \varkappa l \leqslant \sqrt{\pi/2}$. It splits into three sequences for $\varkappa l = \sqrt{\pi/2}$.

One of these corresponds to the following motion: recession from the center, return to the initial point, incidence on the center, and again return to the initial position. These modes are nonsymmetric.

Modes of the second sequence are obtained from modes of the first by a mirror transformation from the middle, and correspond to incidence on the center, return, recession, and again return.

The third sequence contains symmetric modes. Recession from the center, return, incidence, return, again recession, and a final return corresponds to it.

These series of modes exist for all $\varkappa l > \sqrt{\pi/2}$. For $\varkappa l = \sqrt{2\pi}$ there is possible motion when the body having started to move at a zero initial velocity collides twice with the center. The corresponding mode for $\varkappa l > \sqrt{2\pi}$ bifurcates into four new sequences of modes. For $\varkappa l = 3\sqrt{\pi/2}$ four more branches are generated, etc. The number of such branches is infinite.

For $\varkappa l \to 0$ the sequence of modes with a cusp joins the upper half-branch of the sequence (5.9), however their limit modes differ in that the repelling strings interlace (the "upper" string passes under the "lower" at the cusp point). This latter is understandable since motion with a collision is the limit case of motions enveloping the center, and it should be considered that a value $u = -0$ is reached upon impact, i.e. the body "sets" behind the center.

Nonplanar modes cannot be considered here. Let us just note that they are known to exist. Thus modes corresponding to a circular orbit are found by elementary means. Hence $u = \cos\tau$, $w = \pm\sin\tau$, $\varkappa l = 2\pi$, 4π, etc., and the form of the string is a helical line having n curls for $\varkappa l = 2\pi n$. Such modes exist in pairs: as right-hand and left-hand spirals, which corresponds to two directions of body rotation around the center. Seeking the remaining modes is substantially more complex than determining the periodic motions in the Newtonian potential case, for example. This is seen at least from the fact that an unrealizable quadrature will replace the equations of the conic sections.

BIBLIOGRAPHY

1. Appel, P., Figures of Equilibrium of a Rotating Homogeneous Fluid. Moscow-Leningrad, ONTI, 1936.
2. Hardy, G. G., Littlewood, D. E. and Polya, G., Inequalities (Russian translation), Moscow, IIL, 1948.
3. Bellman, R., Theory of Stability of Solutions of Differential Equations. (Russian translation). Moscow, IIL, 1954.

Translated by M. D. F.

PROPAGATION OF A SHOCK WAVE IN A CHANNEL WHEN SHOCK-COMPRESSED GAS INTERACTS WITH A NONHOMOGENEOUS MAGNETIC FIELD

PMM Vol. 34, №4, 1970, pp. 672-684

A. B. VATAZHIN
(Moscow)

(Received March 30, 1970)

Unsteady flow of a conducting gas under shock wave conditions in channels of various magnetohydrodynamic devices was investigated in several recent papers (see for example [1, 2]). Most of them assume that the electrical current distribution in the gas behind the shock wave is one-dimensional, and that it is controlled by the conditions of current closure in the external electrical circuit that connects the electrodes at the channel walls.

However, in real channels there are always regions where the magnetic field is nonhomogeneous and where the channel walls are nonconducting. As a rule, these regions coincide with the end zones of the external magnetic field. Behind the shock wave passing through the end zones in the gas there are closed electrical currents whose intensity depends on the position of the shock front. These two-dimensional currents interact with the magnetic field and cause perturbations which catch up with the shock wave and change its velocity.

Terminal effects in steady magnetohydrodynamic flows have been investigated for a long time (see for example [3]) but their influence on the unteady gas flow has not yet been solved definitely. Among the papers devoted to this subject matter are two experimental studies [4, 5] which indicate that a substantial change occurs in the velocity of the plasma front in nonhomogeneous magnetic field, and that this effect is related to the emergence of closed-current zones in the plasma.